A GENERAL FORMULATION FOR LARGE STRAINS HYPERELASTIC TRUSSES

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Abstract. *Within the large strains regime, manifold theory is used in order to define the truss kinematics in the most fundamental way. The complete deformation gradient of the truss is revealed and consequently, Hill's large strain measures are computed. It is shown that for an isotropic, hyperelastic truss, stress – strain conjugacy depends on the reckoned truss volume. If conjugacy is defined in the current truss volume, then even for the simplest linear hyperelastic constitutive model, the truss's structural behavior depends on its cross sectional kinematics. This shows that the large strain hyperelastic truss is at most a semi – one – dimensional problem. Thus, it is seen that a general formulation of the large strains hyperelastic truss requires precise definitions of its kinematics, kinetics, constraints, constitutive laws and notions of conjugacy and description. These issues are comprehensively dealt with in the present paper.*

1 INTRODUCTION

The mental prototype of a truss consists of a straight, cylindrical rod capable of carrying only an axial force and whose length greatly surpasses its cross sectional dimensions. Our mind, tends to abstract this rod as being one dimensional, eliminating its cross sectional extent and consequent three dimensional character. This idealization, is apparent in most large deformation or large strains theories of trusses, wherein the truss is treated as a purely one dimensional continuum [1, 6, 10 and 14], with no consideration of its cross sectional behavior.

Visualize a rod with a changing length, moving in space such that it always remains straight, while its cross section changes in shape and size. Could this rod constitute a truss? The only difference of this picture from the aforementioned mental prototype is the inclusion of the cross sectional kinematics. However, there need to be some restrictions on the possible cross sectional changes of this rod if it is to describe a 'truss', carrying only an axial force. It is found, that the fundamentals of Continuum Mechanics and the assumptions used in the geometrical construction of the truss's deformation gradient from first principles (using manifold theory), dictate the necessary ingredients for an in – depth answer to this question. In the following, this answer is given in particular, for the special case of a linear, isotropic, hyperelastic truss.

More specifically, an attempt is made to describe the truss as a three dimensional continuum. It is found that this is not as simple as one might imagine, even for the linear, isotropic, hyperelastic case, since the truss continuum has to be augmented with numerous constraints in order to consistently represent a 'truss' in a kinematic and kinetic point of view. These kinematic and kinetic requirements enforce a coupling of the constitutive model with the cross sectional kinematics. For the special case of a linear, isotropic, hyperelastic constitutive model, it is found that this coupling is existent only if conjugacy is reckoned with respect to current volume; hence the truss can be said to be a semi – one dimensional continuum. If conjugacy is defined with respect to the reference volume (again resulting in a linear, isotropic, hyperelastic model but this time with different material constants) this coupling 'disappears' and the truss becomes truly one dimensional.

In many references [4 – 6 and 18], the effect of the truss's changing cross sectional area during its motion is dealt with in an *ad hoc* manner, that is not formalized or supported on firm theoretical grounds, irrespective of the end result's correctness. In [2, 7, 9 and 12], the coupling of the cross sectional kinematics with the constitutive model is derived for various hyperelastic models with reference to a uniaxial rod under tension, but without an explicit investigation of its origins or the geometrical implications it has to the truss's cross section. In this paper, the geometric, kinematic and kinetic origins of this coupling are uncovered, via the construction of the truss's complete deformation gradient and the comprehensive investigation of stress – strain conjugacy.

In [3], it is shown that there is a necessary dependence of the various large strain measures to the constitutive constants for the general, linear – hyperelastic, three – dimensional continuum. However, in the case of the three – dimensional truss continuum, this dependence becomes a coupling between that cross sectional kinematics and the constitutive model that is also affected by the choice of volume used for conjugacy definition.

The end product of this investigation is the derivation of closed form relations (representing the relation of global nodal forces versus global nodal displacement) for all possible strain measures, to be used in Finite Element Analyses of large strains, isotropic, linear hyperelastic trusses allowing or not for volume change. Also, the methodology used may be applied to other more complicated elements such as cables, membranes, beams, shells etc.

2 KINEMATICS

2.1 Embeddings, coordinate frames and transformations between frames and configurations

We view the truss as a three – dimensional differentiable manifold, whose elements are called *particles* [15]. To every particle P we assign in a one – to – one manner, a triplet of real numbers (s_1, s_2, s_3) as identification parameters. We describe the longitudinal direction via the s_1 parameter, with $s_1 \in [0, L]$ and the cross – section via the s_2, s_3 parameters. We postulate the existence of a one parameter family $A(s_1) = A$ of bounded areas in R^2 that describes the cross section as we move along the coordinate s_1. We assume that the one dimensional s_1 submanifold of the truss manifold is globally homeomorphic to R. Under these assumptions, the notion of length in s_1 and area in $A(s_1)$, can be defined [13].

In order to make observations on the truss, we perform embeddings of the truss manifold within the three dimensional Euclidean space. Any of the infinitely many possible embeddings of the truss manifold corresponds to a distinct configuration of the truss, occupying some portion of the Euclidean space. The various possible configurations are denoted by ${}^{m}C$, where m takes the values 0, 1, 2…, assumed to correspond to some possible labeling of the configurations, necessary for discussion purposes. We postulate the existence of some natural reference configuration corresponding to $m = 0$, denoted by ${}^{0}C$. This configuration is supposed to correspond to the natural state that the truss would have if it were completely isolated from its surrounding.

By an embedding, we mean the one – to – one assignment of coordinates to each parameter of the manifold, so that the truss does not self – intersect. These coordinates correspond to some selected fixed frame in space and it follows that after the embedding, every particle of the manifold corresponds to a unique triplet of Euclidean space coordinates relative to the selected reference frame. Let us denote the fixed orthonormal frame of reference in the Euclidean space by $X \equiv \{X_1, X_2, X_3\}$ meaning that X_1, X_2, X_3 are the corresponding axes with unit vectors $\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_2, \hat{\mathbf{X}}_3$ respectively. Then, mathematically, an embedding is described as $X_q = X_q(s_r)$, where q, r = 1, 2, 3. Acceptable homogeneous embeddings of the s_1 truss submanifold in the Euclidean space are assumed to correspond to straight – line – segments having length ${}^{m}L$, while acceptable homogeneous embeddings of the s_2 – s_3 submanifold correspond to bounded plane surfaces having area ${}^{m}A$ and being normal to the embedded s_1 line.

The embedding of the s_1 submanifold is trivial, since we get a straight line segment with length ${}^{m}L$, determined solely from the coordinates of its ends, and defining a local ${}^{m}x_1$ axis, with a unit basis vector ${}^{m}\hat{\mathbf{x}}_1$. In order to get a plane – area embedding of the s_2 – s_3 submanifold, we need two linearly independent vectors (not necessarily orthogonal), call them ${}^{m}\mathbf{x}_2, {}^{m}\mathbf{x}_3$ that span this plane area. For this plane area to be orthogonal to the s_1 – line, these two vectors have to be orthogonal with ${}^{m}\hat{\mathbf{x}}_1$, so they too define two local axes, ${}^{m}x_2$ and ${}^{m}x_3$ respectively with the corresponding unit base vectors.

The union of the two submanifolds defined above gives us the entire embedded truss manifold.

In order to construct the deformation gradient of the truss, we need to consider the motion of the truss between two distinct configurations, one taken as the current configuration, say ${}^{m}C$ and the other taken as an intermediate configuration ${}^{n}C$, as shown in figure 1.

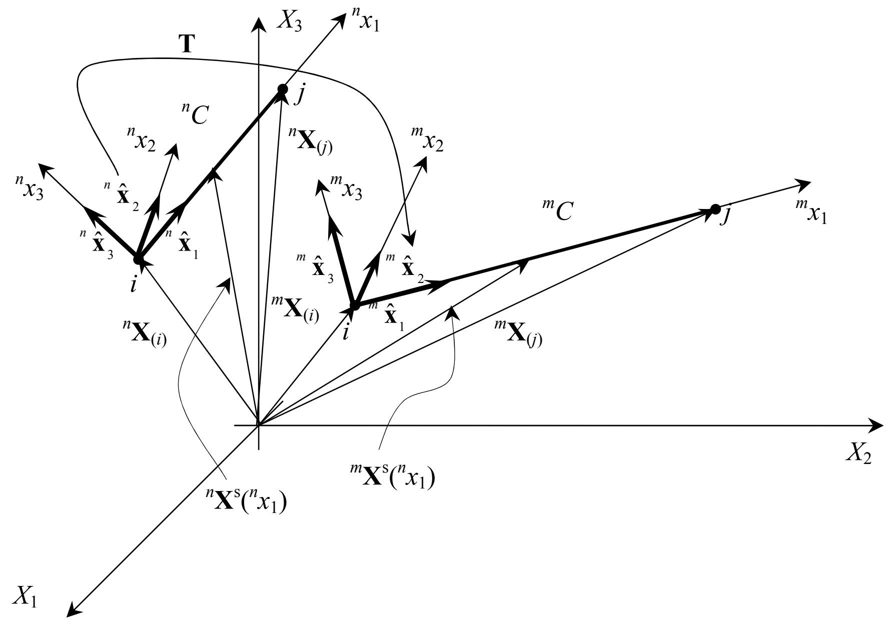

Figure 1. The truss's motion from ${}^{n}C$ to ${}^{m}C$ (the cross sectional area is not shown).

The global X frame is shown along with two local frames, ${}^{m}x$ and ${}^{n}x$, one in each respective configuration (${}^{m}C$ and ${}^{n}C$) with their corresponding basis vectors. $\mathbf{X}_{(i)}$ and $\mathbf{X}_{(j)}$ are the position vectors of the end points of the truss and a left superscript denotes the configuration in question. Similarly, $\mathbf{X}^{s}$ are position vectors of points on the embedded s_1 – line. For the cross section spanning vectors, we have ${}^{p}\mathbf{x}_r = {}^{p}\lambda_r\, {}^{p}\hat{\mathbf{x}}_r$, where r = 2, 3 and $p = m$ or n. Clearly, ${}^{p}\lambda_r$ refers to the norm of each respective vector used to describe the embedded cross section of the truss. **T** symbolizes the transformation from ${}^{m}C$ to ${}^{n}C$. From the figure, one can see that this transformation involves a change in the length of the truss as well as a rigid body rotation. Although absent from the figure for simplicity, one can mentally visualize the changing of the truss's cross section in going from ${}^{m}C$ to ${}^{n}C$.

It is **T** that we want to determine first, so consider a free vector **a**. We denote this vector's representation in some f frame, by $\mathbf{a}\big|_f$. Then we can write: $\mathbf{a}\big|_f = \mathbf{R}_f \cdot \mathbf{a}\big|_X$, with $\mathbf{R}_f$ being the transformation matrix relating the f and X components of **a**. Unless f is an orthogonal frame, $\mathbf{R}_f$ is not orthogonal. With this notation at hand and using figure 1 (not restricted to points on the embedded s_1 – line but including the cross section), it is found that we may write:

$$\left({}^{m}\mathbf{X}(s_1,s_2,s_3) - {}^{m}\mathbf{X}_{(i)} - {}^{m}\mathbf{X}^{A}_{(c)}\right)\Big|_f = {}^{m}_{n}\mathbf{F}_{{}^{m}x{}^{n}x}\Big|_{fg} \cdot \left({}^{n}\mathbf{X}(s_1,s_2,s_3) - {}^{n}\mathbf{X}_{(i)} - {}^{n}\mathbf{X}^{A}_{(c)}\right)\Big|_g \tag{1a}$$

where

$${}^{m}_{n}\mathbf{F}_{{}^{m}x{}^{n}x}\Big|_{fg} = \mathbf{R}_f \cdot {}^{m}_{n}\mathbf{F}_{{}^{m}x{}^{n}x}\Big|_X \cdot \mathbf{R}_g^{-1} \tag{1b}$$

$${}^{m}_{n}\mathbf{F}_{{}^{m}x{}^{n}x}\Big|_X = \mathbf{R}^{-1}_{{}^{m}x} \cdot {}^{m}_{n}\mathbf{\Lambda}_{{}^{m}x{}^{n}x} \cdot \mathbf{R}_{{}^{n}x} \tag{1c}$$

$${}^{m}_{n}\mathbf{\Lambda}_{{}^{m}x{}^{n}x} = \begin{bmatrix} {}^{m}_{n}\lambda & 0 & 0 \\ 0 & {}^{m}_{n}\lambda_2 & 0 \\ 0 & 0 & {}^{m}_{n}\lambda_3 \end{bmatrix} \tag{1d}$$

$${}^{m}_{n}\lambda = \frac{{}^{m}L}{{}^{n}L},\quad {}^{m}_{n}\lambda_2 = \frac{{}^{m}\lambda_2}{{}^{n}\lambda_2},\quad {}^{m}_{n}\lambda_3 = \frac{{}^{m}\lambda_3}{{}^{n}\lambda_3} \tag{1e}$$

Note that f and g are arbitrary frames used for the representations. The vectors on the left and right hand sides of (1a), denote the relative position vectors of points of the embedded truss in ${}^{m}C$ and ${}^{n}C$ respectively. These vectors are relative to the point with (s_1, s_2, s_3) = (0, 0, 0) having position vectors ${}^{m}\mathbf{X}_{(i)} + {}^{m}\mathbf{X}^{A}_{(c)}$ and ${}^{n}\mathbf{X}_{(i)} + {}^{n}\mathbf{X}^{A}_{(c)}$ respectively, with ${}^{m}\mathbf{X}^{A}_{(c)}$ and ${}^{n}\mathbf{X}^{A}_{(c)}$ being the points where the s_1 – line impinges the cross section. The **F** in (1) is **T** of figure 1. The transformation in (1) is known as a point transformation [7] or a point mapping [11] that relates position vectors of the truss in ${}^{m}C$ to those in ${}^{n}C$.

2.2 The deformation gradient, its polar decomposition and Hill's strain measures and their rates

Equation (1) tells us how points transform, but we need to know how line elements transform in order to uncover the ingredients of the truss's motion [2, 7, 9 11 and 15]. A line element of the embedded truss is a vector bound at some given point that connects this point to some other point of the truss that is infinitesimally close to it, the collection of all such infinitesimally – close – points forming the tangent space of the embedded truss [7, 11]. By taking the derivative of (1), and keeping in mind that **F** is a constant, it can easily be shown that **F** of (1) is also the linear mapping transforming line elements in ${}^{m}C$ to line elements in ${}^{n}C$. In other words, **F** of (1) is the deformation gradient of the truss [7, 11]. The truss kinematics amount to homogeneous deformations i.e. straight line segments and plane areas remain straight and plane respectively during the motion of the truss. Hence, material and tangent line elements transform via the same linear mapping **F** and the tangent space of the truss is said to coincide with the Euclidean space occupied by the truss itself [7].

Having the deformation gradient of the truss, we proceed to obtain its polar decomposition. According to the polar decomposition theorem, we have that [2, 7, 9, 11 and 17]:

$${}^{m}_{n}\mathbf{F}_{{}^{m}x{}^{n}x} = {}^{m}_{n}\mathbf{R}_{{}^{m}x{}^{n}x} \cdot {}^{m}_{n}\mathbf{U}_{{}^{m}x{}^{n}x} = {}^{m}_{n}\mathbf{V}_{{}^{m}x{}^{n}x} \cdot {}^{m}_{n}\mathbf{R}_{{}^{m}x{}^{n}x} \tag{2}$$

Where **R** is an orthogonal matrix representing a rigid body rotation and **U**, **V** are the right and left stretch matrices respectively, which are both symmetric and positive definite. In this paper, we are interested in the conditions that make **U** and **V** intrinsically diagonal, because of the simplicity involved and the analytical form of the equations sought for. Also, the cases where **U** and **V** are intrinsically diagonal possess certain symmetry (equation (4) below) of the Eulerian and the Lagrangian counterparts [17]. By pure mathematical procedures and following the algorithm of polar decomposition [11], we get that either of the following conditions should necessarily hold for **U** and **V** to be diagonal:

$$ {}^m_n\lambda_2 = {}^m_n\lambda_3 \;\&\; {}^m\hat{\mathbf{x}}_2\cdot{}^m\hat{\mathbf{x}}_3 = {}^n\hat{\mathbf{x}}_2\cdot{}^n\hat{\mathbf{x}}_3 \tag{3a} $$

$$ {}^m\hat{\mathbf{x}}_2\cdot{}^m\hat{\mathbf{x}}_3 = {}^n\hat{\mathbf{x}}_2\cdot{}^n\hat{\mathbf{x}}_3 = 0 \tag{3b} $$

Note that $(3a)_2$ tells us that the vectors spanning the truss's cross section should have the same angle between them, in all configurations, while at the same time, by $(3a)_1$, they should also have equal ratios of corresponding norms. (3b) tells that if the cross section spanning basis is orthogonal, then no restriction on the norms is at hand. However, (3) is not all it takes for **U** and **V** being diagonal. When (3a) holds true, the subscript ${}^mx\sim{}^nx$ will be used, whereas when (3b) holds true the subscript x will be used to denote this fact. Thus, we can say that we are looking for the conditions that will make ${}^m_n\mathbf{U}_{{}^mx\sim{}^nx}, {}^m_n\mathbf{V}_{{}^mx\sim{}^nx}$ and ${}^m_n\mathbf{U}_x, {}^m_n\mathbf{V}_x$ diagonal. Clearly, these conditions have to do with the frames used in their representations. In the case (3a) that the ${}^mx$ and ${}^nx$ frames are not orthogonal, we can get unique orthogonal local frames, denoted by ${}^mx_o$ and ${}^nx_o$ respectively, corresponding to the orthogonal rotation matrix extracted by polarly decomposing the non – orthogonal transformation matrix relating ${}^mx$ and ${}^nx$ to X. With all this, it follows that:

$$ {}^m_n\mathbf{U}_{{}^mx\sim{}^nx}\Big|_{{}^nx_o} = {}^m_n\mathbf{V}_{{}^mx\sim{}^nx}\Big|_{{}^mx_o} = {}^m_n\mathbf{\Lambda}_{{}^mx\sim{}^nx} \tag{4a} $$

$$ {}^m_n\mathbf{U}_x\Big|_{{}^nx} = {}^m_n\mathbf{V}_x\Big|_{{}^mx} = {}^m_n\mathbf{\Lambda}_x \tag{4b} $$

Where, each $\mathbf{\Lambda}$ in (4) is obtained from combining (1d) with (3). Notice that $\mathbf{\Lambda}$ in (4a) has only two distinct eigenvalues. **U** is known to be a Lagrangian tensor, whereas **V** is Eulerian, so according to [17], (4) tells us that the Lagrangian counterpart of ${}^m_n\mathbf{V}_x\big|_{{}^nx}$ is the identical to the Eulerian counterpart of ${}^m_n\mathbf{U}_x\big|_{{}^mx}$, while at the same time, it is also the Eulerian counterpart of ${}^m_n\mathbf{V}_x\big|_{{}^mx}$, and similarly for ${}^m_n\mathbf{V}_{{}^mx\sim{}^nx}$. This is because the transformation matrix relating ${}^mx$ and ${}^nx$ is the same as the rotation matrix of the polar decomposition in (2).

The expression in (4) could be made more general by using two representation frames for **U** and two for **V**. One would be the same as in (4) but the other would be arbitrary. This generality – that could also be present elsewhere (i.e. dealing with the non – diagonal cases) – is not really necessary, since the basic aims of the paper are fulfilled by this simpler approach.

The important thing about the polar decomposition is that **U** and **V** contain all the information about the deformation of the truss which is of interest to continuum mechanics [7], whereas the rigid body portion of the motion, given by **R** is surpassed. Using **U** and **V** of (4), we can easily obtain the Lagrangian **E** and Eulerian **ε**, Hill's strain measures [2, 4, 7, 9, 15, and 17], that measure the deformation of the truss experienced in three independent directions while going from ${}^nC$ to ${}^mC$.

$$ {}^m_n\mathbf{E}^{(k)}_{{}^mx\sim{}^nx}\Big|_{{}^nx_o} = {}^m_n\boldsymbol{\varepsilon}^{(k)}_{{}^mx\sim{}^nx}\Big|_{{}^mx_o} = \frac{1}{k}\left({}^m_n\mathbf{\Lambda}^k_{{}^mx\sim{}^nx} - \mathbf{I}\right) \tag{5a} $$

$$ {}^m_n\mathbf{E}^{(k)}_x\Big|_{{}^nx} = {}^m_n\boldsymbol{\varepsilon}^{(k)}_x\Big|_{{}^mx} = \frac{1}{k}\left({}^m_n\mathbf{\Lambda}^k_x - \mathbf{I}\right) \tag{5b} $$

Where, $k = \ldots -2, -1, 0, 1, 2 \ldots$ and **I** is the identity matrix. Note that for $k = 0$, we get Hencky's or Logarithmic strain measures [15, 16]. Now, the rates of the strain measures in (5) are needed later on, so we let:

$$ {}^p\mathbf{\Lambda}_{{}^px} = \begin{bmatrix} {}^p\lambda & 0 & 0 \\ 0 & {}^p\lambda_2 & 0 \\ 0 & 0 & {}^p\lambda_2 \end{bmatrix} \text{ and } {}^p\mathbf{\Lambda}_x = \begin{bmatrix} {}^p\lambda & 0 & 0 \\ 0 & {}^p\lambda_2 & 0 \\ 0 & 0 & {}^p\lambda_3 \end{bmatrix}, \text{ with } {}^p\lambda = \frac{{}^pL}{L} \tag{6} $$

With (6), we can say that:

$$ {}^m_n\mathbf{\Lambda}_{{}^mx\sim{}^nx} = {}^m\mathbf{\Lambda}_{{}^mx}\cdot{}^n\mathbf{\Lambda}^{-1}_{{}^nx} \text{ and } {}^m_n\mathbf{\Lambda}_x = {}^m\mathbf{\Lambda}_x\cdot{}^n\mathbf{\Lambda}^{-1}_x \tag{7} $$

Note that there is no time dependence of the intermediate ${}^nC$ configuration since it is considered to be a reference configuration [15]. With all these and using (5) and (7), we get that:

$$ {}^{m}_{n}\dot{\varepsilon}^{(k)}_{{}^{m}x\sim{}^{n}x}(t)\Big|_{{}^{m}x_o} = {}^{m}\mathbf{d}_{{}^{m}x\sim{}^{n}x}(t)\Big|_{{}^{m}x_o} \cdot {}^{m}\Lambda^{k}_{{}^{m}x}(t) \cdot {}^{n}\Lambda^{-k}_{{}^{n}x} \text{ and } {}^{m}_{n}\dot{\varepsilon}^{(k)}_{x}(t)\Big|_{{}^{m}x} = {}^{m}\mathbf{d}_{x}(t)\Big|_{{}^{m}x} \cdot {}^{m}\Lambda^{k}_{x}(t) \cdot {}^{n}\Lambda^{-k}_{x} \tag{8} $$

Where, ${}^{m}\mathbf{d}_{{}^{m}x\sim{}^{n}x}(t)\Big|_{{}^{m}x_o} = {}^{m}\dot{\Lambda}_{{}^{m}x}(t) \cdot {}^{m}\Lambda^{-1}_{{}^{m}x}(t)$ and ${}^{m}\mathbf{d}_{x}(t)\Big|_{{}^{m}x} = {}^{m}\dot{\Lambda}_{x}(t) \cdot {}^{m}\Lambda^{-1}_{x}(t)$ are known as the corresponding Eulerian stretching or rates of deformation tensors [2, 4, 7, 9 and 16]. Clearly, in (8), we get that for $k = 0$, the material rate of the Eulerian Logarithmic strain coincides with the Eulerian rate of deformation. This is also true for their Lagrangian counterparts. The reason for this equality is the fact that the principal directions of the left and right stretch tensors remain fixed (in a corotational sense) during the truss's motion [8]. It can also be shown, that the results obtained in [17] hold true for the truss in greatly simplified forms.

In the rest of the development, we will only include the case of orthogonal cross section spanning vectors, i.e. equation (3b), since all the results following (3a) are completely analogous to those of (3b), the only difference being that (3a) has two (instead of three) distinct eigenvalues of the stretch tensors.

3 KINETICS AND STRESS – STRAIN CONJUGACY

3.1 The Cauchy stress and the kinetic definition of a truss

It is well known that the state of stress at a point of a given continuum (assuming no concentrated couples) is represented by the Cauchy stress tensor ${}^{m}\boldsymbol{\tau}$ [2, 4, 7, 9, 11and 15]. The truss carries only a constant axial load, so this requirement along with the permissible loading of the truss (only nodal) results in a constant stress tensor throughout the truss's volume. The kinetic definition of the truss that is consistent with its boundary conditions amounts to the Cauchy stress tensor of the truss being given by:

$$ \left({}^{m}\boldsymbol{\tau}_{x}\Big|_{{}^{m}x} \right)_{11} = \begin{bmatrix} {}^{m}t_{x11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{9} $$

In other words, we have that the stress tensor of the truss has only one non vanishing constant component when it is represented in the current local frames. Since the truss always remains straight and only carries an axial force, truss buckling can not be accounted for directly, but only through constitutive modeling [1].

3.2 Stress power and stress – strain conjugacy for an isotropic hyperelastic truss

For every material body, the rate of work of the external forces equals the sum of the temporal change of the kinetic energy and the stress power [7]. When the material body has no memory (i.e. its state depends only on the state of deformation), while its stress power is equal to the strain energy rate in the absence of any heat exchange with the environment, it is called a hyperelastic body [7]. Assuming conservative loads acting on this body, the conservation of mechanical energy is satisfied [7]. We assume that the truss is such a hyperelastic body acted upon by conservative nodal loads. It follows [7, 9 and 15 - 17], that the strain energy rate, ${}^{m}\dot{U}_{x}$ of the truss is:

$$ {}^{m}\dot{U}_{x} = \iiint_{{}^{m}V} {}^{m}\dot{u}_{x,{}^{m}V} d^{m}V = \iiint_{{}^{m}V} tr\left[{}^{m}\boldsymbol{\tau}_{x} \cdot {}^{m}\mathbf{d}_{x} \right] d^{m}V = \iiint_{{}^{n}V} {}^{m}\dot{u}_{x,{}^{n}V} d^{n}V \tag{10} $$

Where ${}^{m}V$, ${}^{n}V$ is the current and reference truss volume respectively and tr is the trace operator. The current and reference volume are related by ${}^{m}V = \det\left[{}^{m}_{n}\Lambda_{x}\right]{}^{n}V$, i.e. via $\det\left[{}^{m}_{n}\Lambda_{x}\right] = {}^{m}_{n}\lambda \, {}^{m}_{n}\lambda_{2} \, {}^{m}_{n}\lambda_{3}$, the Jacobian [2, 4, 7, 9, 11 and 15] of the deformation. ${}^{m}\dot{u}_{x,{}^{m}V} = tr\left[{}^{m}\boldsymbol{\tau}_{x} \cdot {}^{m}\mathbf{d}_{x} \right]$, ${}^{m}\dot{u}_{x,{}^{n}V} = \det\left[{}^{m}_{n}\Lambda_{x}\right] {}^{m}\dot{u}_{x,{}^{m}V}$ is the strain energy density rate per unit current and reference volume respectively. From (10), the Cauchy stress is conjugate to ${}^{m}\mathbf{d}_{x}$, with respect to the current volume, whereas the Kirchhoff stress, ${}^{m}\boldsymbol{\sigma}_{x} = \det\left[{}^{m}_{n}\Lambda_{x}\right] {}^{m}\boldsymbol{\tau}_{x}$, is conjugate to ${}^{m}\mathbf{d}_{x}$, with respect to the reference volume [2, 7 and 17].

It can be shown that the stress tensor in (10) should be a function of the stretch tensor if one wants to have a complete mathematical formulation of the state equations [7, 15]. Assuming we are dealing with an isotropic, hyperelastic body, the Cauchy stress tensor is given by [7, 15]:

$$ {}^{m}\boldsymbol{\tau}_{x} = {}^{m}\boldsymbol{\tau}_{x}\left({}^{m}\Lambda_{x} \right) \tag{11} $$

By (11) in (10), we have (considering the invariance of the trace operator) that ${}^{m}\dot{u}_{x,{}^{m}V} = tr\left[{}^{m}\boldsymbol{\tau}_{x}\left(\cdot {}^{m}\Lambda_{x} \right)\Big|_{{}^{m}x} \cdot {}^{m}\dot{\Lambda}_{x} \cdot {}^{m}\Lambda^{-1}_{x} \right]$. Now, we can say that ${}^{m}\dot{u}_{x,{}^{m}V} = tr\left[{}^{m}\boldsymbol{\tau}_{x}\left(\cdot \ln\left({}^{m}\Lambda_{x}\right)\right)\Big|_{{}^{m}x} \cdot d\ln\left({}^{m}\Lambda_{x}\right)/dt \right] = tr\left[\partial {}^{m}u_{x,{}^{m}V} / \partial \ln\left({}^{m}\Lambda_{x}\right) \cdot d\ln\left({}^{m}\Lambda_{x}\right)/dt \right]$. To make the strain energy density rate integrable (i.e. an exact differential), we get from all this that the Cauchy stress should be derivable from a potential [7], i.e.:

$$ {}^{m}\boldsymbol{\tau}_{x}\Big|_{{}^{m}x} = \frac{\partial\, {}^{m}u_{x,{}^{m}V}}{\partial \ln\left({}^{m}\boldsymbol{\Lambda}_{x}\right)} \tag{12} $$

In a similar manner, it follows that for the Kirchhoff stress tensor, (12) applies by simply replacing **σ** for **τ** while also using the strain energy density rate per unit reference instead of current volume.

Using (8), (11) and (12) in (10), relations completely analogous to (12) can easily be derived, but involving derivatives with respect to the generalized strain measures of (5):

$$ {}^{m}_{n}\boldsymbol{\tau}^{(k)}_{x}\Big|_{{}^{m}x} = \frac{\partial\, {}^{m}u_{x,{}^{m}V}}{\partial\, {}^{m}_{n}\boldsymbol{\varepsilon}^{(k)}_{x}\Big|_{{}^{m}x}} \text{ and } {}^{m}_{n}\boldsymbol{\sigma}^{(k)}_{x}\Big|_{{}^{m}x} = \frac{\partial\, {}^{m}u_{x,{}^{n}V}}{\partial\, {}^{m}_{n}\boldsymbol{\varepsilon}^{(k)}_{x}\Big|_{{}^{m}x}} \tag{13} $$

It follows, that the stress measures in $(13)_1$ and $(13)_2$, given in the following (equation (14)), are conjugate to the corresponding strain measures found in (5b), with respect to the current and to the reference volume respectively.

$$ {}^{m}_{n}\boldsymbol{\tau}^{(k)}_{x} = {}^{m}\boldsymbol{\Lambda}^{-k}_{x}\cdot{}^{n}\boldsymbol{\Lambda}^{k}_{x}\cdot{}^{m}\boldsymbol{\tau}_{x} \text{ and } {}^{m}_{n}\boldsymbol{\sigma}^{(k)}_{x} = \det\left[{}^{m}_{n}\boldsymbol{\Lambda}_{x}\right]{}^{m}_{n}\boldsymbol{\tau}^{(k)}_{x} \tag{14} $$

Note that for $k = 0$, we find that the Eulerian Logarithmic strain measure is conjugate to the Cauchy stress, with respect to the current volume, whereas it is conjugate to the Kirchhoff stress with respect to the reference volume. Stress – strain conjugacy enables one to consistently relate these measures via constitutive modeling so that specific forms of the strain energy of the truss can be computed.

4 THE SIMPLEST ISOTROPIC, LINEAR – HYPERELASTIC LARGE STRAINS TRUSS

4.1 The simplest isotropic, linear – hyperelastic constitutive model and constitutive transformations

There are numerous functional forms that can be used for the strain energy density, each resulting in a unique constitutive model representing a unique hyperelastic material [7, 15]. The simplest isotropic, linear – hyperelastic model that results is obtained by assuming that:

$$ {}^{m}u_{x,{}^{p}V} = (1/2)\left({}_{n}a^{(k)}_{{}^{p}V} + 2\,{}_{n}b^{(k)}_{{}^{p}V}\right) I^{2}_{(k)} + 2\,{}_{n}b^{(k)}_{{}^{p}V} II_{(k)} + {}_{n}c^{(k)}_{{}^{p}V}\left({}^{m}_{n}\boldsymbol{\varepsilon}^{(k)}_{x}\Big|_{{}^{m}x}\right)_{11} + {}_{n}d^{(k)}_{{}^{p}V} \tag{15} $$

$$ \text{with } I_{(k)} = \left({}^{m}_{n}\boldsymbol{\varepsilon}^{(k)}_{x}\Big|_{{}^{m}x}\right)_{11} + \left({}^{m}_{n}\boldsymbol{\varepsilon}^{(k)}_{x}\Big|_{{}^{m}x}\right)_{22} + \left({}^{m}_{n}\boldsymbol{\varepsilon}^{(k)}_{x}\Big|_{{}^{m}x}\right)_{33} \text{ and} $$

$$ II_{(k)} = (1/2)\left\{ \left({}^{m}_{n}\boldsymbol{\varepsilon}^{(k)}_{x}\Big|_{{}^{m}x}\right)^{2}_{11} + \left({}^{m}_{n}\boldsymbol{\varepsilon}^{(k)}_{x}\Big|_{{}^{m}x}\right)^{2}_{22} + \left({}^{m}_{n}\boldsymbol{\varepsilon}^{(k)}_{x}\Big|_{{}^{m}x}\right)^{2}_{33} - \left[\left({}^{m}_{n}\boldsymbol{\varepsilon}^{(k)}_{x}\Big|_{{}^{m}x}\right)_{11} + \left({}^{m}_{n}\boldsymbol{\varepsilon}^{(k)}_{x}\Big|_{{}^{m}x}\right)_{22} + \left({}^{m}_{n}\boldsymbol{\varepsilon}^{(k)}_{x}\Big|_{{}^{m}x}\right)_{33} \right]^{2} \right\} $$

Being the first and second invariants of the Eulerian strain tensor ${}^{m}_{n}\boldsymbol{\varepsilon}^{(k)}_{x}\Big|_{{}^{m}x}$ and p could be m or n. Note that the constitutive constants used are fourth order tensors [2, 3, 7 and 9], which depend on the strain measure used (superscript), on the reference configuration chosen (left subscript) and on the volume used for conjugacy definition (right subscript). Also, the mixed derivatives of the strain energy density function in (15) are equal, so it constitutes a valid potential function [3], expressed in terms of the principal stretches.

Using (9) and (15) in (13), we find that:

$$ \left({}^{m}_{n}\boldsymbol{\tau}^{(k)}_{x}\Big|_{{}^{m}x}\right)_{11} = {}_{n}E^{(k)}_{{}^{m}V}\left({}^{m}_{n}\boldsymbol{\varepsilon}^{(k)}_{x}\Big|_{{}^{m}x}\right)_{11} + \det\left[{}^{m}_{n}\boldsymbol{\Lambda}^{-1}_{x}\right]\left({}^{n}\boldsymbol{\tau}_{x}\Big|_{{}^{n}x}\right)_{11} \tag{16a} $$

$$ \left({}^{m}_{n}\boldsymbol{\sigma}^{(k)}_{x}\Big|_{{}^{m}x}\right)_{11} = {}_{n}E^{(k)}_{{}^{n}V}\left({}^{m}_{n}\boldsymbol{\varepsilon}^{(k)}_{x}\Big|_{{}^{m}x}\right)_{11} + \left({}^{n}\boldsymbol{\tau}_{x}\Big|_{{}^{n}x}\right)_{11} \tag{16b} $$

$$ {}^{m}A = {}^{n}A\left[1 + {}_{n}v^{(k)}_{{}^{p}V} - {}_{n}v^{(k)}_{{}^{p}V}\,{}^{m}L^{k}\,{}^{n}L^{-k}\right]^{2/k} \tag{16c} $$

Where we defined:

$$ {}_{n}v^{(k)}_{{}^{p}V} = \frac{{}_{n}a^{(k)}_{{}^{p}V}}{2\left({}_{n}a^{(k)}_{{}^{p}V} + {}_{n}b^{(k)}_{{}^{p}V}\right)}, \qquad {}_{n}E^{(k)}_{{}^{p}V} = \frac{{}_{n}b^{(k)}_{{}^{p}V}\left(3\,{}_{n}a^{(k)}_{{}^{p}V} + 2\,{}_{n}b^{(k)}_{{}^{p}V}\right)}{{}_{n}a^{(k)}_{{}^{p}V} + {}_{n}b^{(k)}_{{}^{p}V}} \tag{17} $$

Note that (16c) is a kinematic constraint for the truss's cross section, coming from the type of constitutive model used and the truss's boundary conditions and that ${}^{n}\boldsymbol{\tau}_{x}$ is the stress in the reference ${}^{n}C$ configuration. The transformations of the constitutive constants in (17) can be shown to be:

$$ {}_{r}E^{(k)}_{{}^{r}V} = {}^{m}_{r}\lambda\, {}^{m}_{r}\lambda_2^{2}\, {}_{r}E^{(k)}_{{}^{m}V}, \quad {}_{r}E^{(k)}_{{}^{m}V} = \left({}^{n}\lambda^{-2k}\, {}^{r}\lambda^{2k}\right)_{n}E^{(k)}_{{}^{m}V}, \quad {}_{r}E^{(k)}_{{}^{r}V} = {}^{n}_{r}\lambda\, {}^{n}_{r}\lambda_2^{2}\left({}^{n}\lambda^{-2k}\, {}^{r}\lambda^{2k}\right)_{n}E^{(k)}_{{}^{n}V} \tag{18a} $$

$$ {}_{r}\nu^{(k)}_{{}^{r}V} = {}_{r}\nu^{(k)}_{{}^{m}V}, \quad {}_{r}\nu^{(k)}_{{}^{m}V} = {}^{n}\lambda^{-k}\, {}^{r}\lambda^{k}\, {}^{n}\lambda_2^{k}\, {}^{r}\lambda_2^{-k}\, {}_{n}\nu^{(k)}_{{}^{m}V}, \quad {}_{r}\nu^{(k)}_{{}^{r}V} = {}^{n}\lambda^{-k}\, {}^{r}\lambda^{k}\, {}^{n}\lambda_2^{k}\, {}^{r}\lambda_2^{-k}\, {}_{n}\nu^{(k)}_{{}^{n}V} \tag{18b} $$

In (16a) and (16b), we see that different E's are involved, depending on the volume used for conjugacy definition. It follows, that these relations are amenable to experimental observation and verification.

4.3 The strain energy of the truss and closed form relations

Integrating (10) over time and volume, using (15) – (17) gives us:

$$ {}^{m}U_{x} = (1/2)\; {}^{m}L\, {}^{n}A\left[1 + {}_{n}\nu^{(k)}_{{}^{p}V} - {}_{n}\nu^{(k)}_{{}^{p}V}\, {}^{m}L^{k}\, {}^{n}L^{-k}\right]^{2/k} {}_{n}E^{(k)}_{{}^{m}V}\left({}^{m}_{n}\boldsymbol{\varepsilon}^{(k)}_{x}\Big|_{{}^{m}x}\right)^{2}_{11} + {}^{n}L\, {}^{n}N_{x}\left({}^{m}_{n}\boldsymbol{\varepsilon}^{(k)}_{x}\Big|_{{}^{m}x}\right)_{11} + {}^{n}U_{x} \tag{19a} $$

$$ {}^{m}U_{x} = (1/2)\; {}^{n}L\, {}^{n}A\, {}_{n}E^{(k)}_{{}^{n}V}\left({}^{m}_{n}\boldsymbol{\varepsilon}^{(k)}_{x}\Big|_{{}^{m}x}\right)^{2}_{11} + {}^{n}L\, {}^{n}N_{x}\left({}^{m}_{n}\boldsymbol{\varepsilon}^{(k)}_{x}\Big|_{{}^{m}x}\right)_{11} + {}^{n}U_{x} \tag{19b} $$

Where ${}^{n}N_{x} = {}^{n}A\left({}^{n}\boldsymbol{\tau}_{x}\Big|_{{}^{n}x}\right)_{11}$ is the axial prestress force present in the reference ${}^{n}C$ configuration. Clearly, (19b) follows from (19a), by using (18a) and (16c).

Observing (19), we see that (19a) applies when (16a) is found true let's say, by an experiment, whereas (19b) applies if (16b) is found to be true. The difference of the two, is that if (16a) is found to be true experimentally, then the strain energy of the truss given in (19a), shows a dependence on the cross sectional kinematics of (16c), i.e. two material constants are necessary for the description of the truss's behavior, E and ν, the second, dictating the manner that the cross section of the truss is changing. On the other hand, if (16b) is found true, there is no cross – sectional dependency of the truss's behavior. Notice that the conjugacy of the stress and strain measures involved in (16a) is with respect to the current truss volume, hence we can say that for the case of the simplest, linear, isotropic – hyperelastic truss, there is a cross sectional dependency of its structural behavior, when conjugacy is reckoned with respect to the current volume. On the other hand, this dependency is absent when conjugacy is defined with respect to the reference volume.

In (19), ${}^{n}U_{x}$ is not known before hand, unless an analysis commencing from some other known state of the truss has been performed. Thus in order to use (19), we have to take n as 0, i.e. choose the reference configuration as the natural reference configuration, wherein ${}^{0}U_{x}$ can be assumed to vanish. Thus, in the following, we will assume that $n = 0$. By (19), one can get closed form relations between the global nodal forces and displacements of the truss. To do this, one simply expresses the length of the truss in the reference and current configurations as:

$$ {}^{0}L = \left[\left({}^{0}X_{1(j)} - {}^{0}X_{1(i)}\right)^{2} + \left({}^{0}X_{2(j)} - {}^{0}X_{2(i)}\right)^{2} + \left({}^{0}X_{3(j)} - {}^{0}X_{3(i)}\right)^{2}\right]^{\frac{1}{2}} \tag{20a} $$

$$ \begin{aligned} {}^{m}L = \Big\{ &\left[\left({}^{0}X_{1(j)} + {}^{m}_{0}U_{1(j)}\right) - \left({}^{0}X_{1(i)} + {}^{m}_{0}U_{1(i)}\right)\right]^{2} + \left[\left({}^{0}X_{2(j)} + {}^{m}_{0}U_{2(j)}\right) - \left({}^{0}X_{2(i)} + {}^{m}_{0}U_{2(i)}\right)\right]^{2} + \\ &+ \left[\left({}^{0}X_{3(j)} + {}^{m}_{0}U_{3(j)}\right) - \left({}^{0}X_{3(i)} + {}^{m}_{0}U_{3(i)}\right)\right]^{2}\Big\}^{\frac{1}{2}} \end{aligned} \tag{20b} $$

Where ${}^{0}X_{q(j)}$, ${}^{0}X_{q(j)}$ (q = 1, 2, 3) are the global coordinates of the truss's endpoints, i and j, while ${}^{m}_{0}U_{1(i)}$, ${}^{m}_{0}U_{1(j)}$ are the truss's global nodal displacements measured from the natural to the current configuration. In a finite element viewpoint, these global nodal displacements are generally the unknowns.

Using (20) in (1d), (1e) and (5), we get the strain – global nodal displacement relations. Using those in (19), with $n = 0$, results in an expression of the truss's strain energy in terms of its global nodal coordinates and displacements, i.e. ${}^{m}U_{x} = {}^{m}U_{x}\left({}^{m}_{0}U_{q(i)}, {}^{m}_{0}U_{q(j)}\right)$. Applying the well known Castigliano's first theorem, we can easily get the global nodal forces $F^{e}_{q(i)}$, $F^{e}_{q(j)}$ as $F^{e}_{q(i)} = \partial^{m}U_{x} / \partial^{m}_{0}U_{q(i)}$ and $F^{e}_{q(j)} = \partial^{m}U_{x} / \partial^{m}_{0}U_{q(j)}$ respectively:

$$ F^{e}_{q(i)} = -\left\{ {}_{0}E^{(k)}_{{}^{m}V}\, {}^{0}A\left[1 + {}_{0}\nu^{(k)}_{{}^{m}V} - {}_{0}\nu^{(k)}_{{}^{m}V}\, {}^{m}L^{k}\, {}^{0}L^{-k}\right]^{2/k} \frac{\left({}^{m}L^{k}\, {}^{0}L^{-k} - 1\right)}{k} + \frac{{}^{0}L\, {}^{0}N_{x}}{{}^{m}L}\right\} \frac{\left(\Delta^{m}_{0}U_{q} + \Delta^{0}X_{q}\right)}{{}^{m}L}, \qquad F^{e}_{q(j)} = -F^{e}_{q(i)} \tag{21} $$

Where $\Delta^{m}_{0}U_{q} = {}^{m}_{0}U_{q(j)} - {}^{m}_{0}U_{q(i)}$, $\Delta^{0}X_{q} = {}^{0}X_{q(j)} - {}^{0}X_{q(i)}$. Using this and (20) in (21) results in closed form expressions relating the global nodal forces and displacements for all possible strain measures, for the case of the simplest linear, isotropic, hyperelastic truss. When conjugacy is reckoned with respect to current volume (as in (21)), these expressions involve terms that stem from the truss cross sectional kinematics, a result showing how the structural behavior of the truss is affected by the manner that its cross section is changing. These relations can be used for the structural analysis of trusses undergoing large hyperelastic strains.

5 CONCLUSIONS

The formulation of a theory for large strains trusses requires a strict application of the principles found in the general theory of Continuum Mechanics. In this paper, we presented the kinematics of a truss which can be used in formulating truss theories of a general type of material. The methodology applied utilizing concepts from manifold theory can also be applied for the formulation of the kinematics of other more complicated structural elements.

As far as linear, isotropic, hyperelastic trusses are concerned, we found that there is a coupling (to be expected physically) of the truss's structural behavior and the way its cross section is changing, particularly when conjugacy is reckoned with respect to the current volume. The development clarified the origins of this coupling and showed the necessity of considering the three dimensional continuum character of the truss when developing formulations involving large strains.

In general, the truss's cross section can be changing along two orthogonal directions or it can change uniformly along two non – orthogonal directions. If the constitutive laws adopted and the truss's boundary conditions are to be consistently satisfied, the cross sectional area of the truss is 'forced' to depend on the constitutive parameter ${}_{n}v_{_{P}V}^{(k)}$ and the truss's length (as implied by (16c)). This result shows, that for linear, hyperelastic trusses, the dependency on constitutive parameters of the functional form of the strain measures produces a kinematic constraint for the cross section (when conjugacy is reckoned with respect to current volume). Thus this simple truss can be said to be at most semi – one dimensional, so quoting [15], "… with little exaggeration, *there are no one - dimensional problems*…".

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